

CLT for continuous random processes under approximations terms.

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ABSTRACT.

We formulate and prove a new sufficient conditions for Central Limit Theorem (CLT) in the space of continuous functions in the terms typical for the approximation theory.

We prove that the *sufficient* conditions for continuous CLT obtained by N.C.Jain and M.B.Marcus are under some natural additional conditions *necessary*.

We provide also some examples in order to show the exactness of obtained results and illustrate briefly the applications in the Monte-Carlo method.

Key words and phrases: Algebraic and trigonometrical approximation, random processes (r.p.) and random fields (r.f.), metric entropy, De la Vallee-Poussin kernel and approximation, Central Limit Theorem in the space of continuous functions, majorizing and minorizing measures, module of continuity, lacunar series, generating functional, uniform equiconvergence.

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1 Notations. Statement of problem.

Let $\eta(t)$, $t \in T = [0, 2\pi]$ be separable centered (zero mean) random process (r.p.) with finite covariation function $R(t, s) = \text{cov}(\eta(t), \eta(s)) = \mathbf{E}\eta(t)\eta(s)$, $\{\eta_i(t)\}$, $i = 1, 2, \dots$ be a sequence of independent copies (identical distributed) of $\eta(t)$. We will denote as usually the probabilistic notions by \mathbf{P} , \mathbf{E} , cov , Var etc.

Let us introduce the following sequence of a random processes:

$$\zeta_n(t) = n^{-1/2} \sum_{j=1}^n \eta_j(t). \quad (1.1)$$

Let also $C(T)$ be a Banach space of all periodical: $f(0) = f(2\pi)$ continuous functions equipped with usually uniform norm $\|f\| = \max_{t \in T} |f(t)|$, $\zeta_\infty(t) = \zeta(t)$ be a

centered separable Gaussian process with at the same covariation function $R(t, s)$. It is clear that the finite-dimensional distributions of r.p. $\zeta_n(t)$ converges as $n \rightarrow \infty$ to the ones for $\zeta_\infty(t)$.

Definition 1.1. We will say as ordinary that the r.p. $\eta(t)$ satisfies the Central Limit Theorem in the space $C(T)$, notation: $\eta(\cdot) \in CLT(C(T))$, if for any continuous bounded functional $F : C(T) \rightarrow R$

$$\lim_{n \rightarrow \infty} \mathbf{E}F(\zeta_n(\cdot)) = \mathbf{E}F(\zeta_\infty(\cdot)).$$

Our purpose in this article is to formulate and prove sufficient conditions for CLT in the space $C(T)$ under the terms which are habitual in the theory of approximation.

This problem has a long history; see e.g. [11], [23], [22], [25], [30], [37]. About applications CLT in the space $C(T)$ in the Monte-Carlo method and in statistics see, e.g. [15], [19], [37].

Indeed, if $\eta(\cdot) \in CLT(C(T))$, then for all positive values u

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\max_{t \in T} |\zeta_n(t)| > u \right) = \mathbf{P} \left(\max_{t \in T} |\zeta_\infty(t)| > u \right).$$

This problem contains as a subproblem the finding of a sufficient condition for continuity with probability one of random processes, Gaussian or not, see [2] - [5], [12] - [14], [16], [20], [21], [24], [22]- [27], [31] - [35], [38] - [41], [43], [47], [49] - [53], [55].

Note that in the articles [11], [23], [22], [30] the CLT is formulated in the so-called *metric entropy* terms, in [11], [20], [37], [43] by means of the notions *majorizing measure*.

The paper is organized as follows. In the next section we formulate and prove a new sufficient conditions for Central Limit Theorem (CLT) in the space of continuous functions. In the third section we prove that the *sufficient* conditions for continuous CLT obtained by N.C.Jain and M.B.Marcus are still under some natural additional conditions *necessary*.

We provide also some examples in the 4th section in order to show the exactness of obtained results and to illustrate briefly the applications in the Monte-Carlo method. The last section contains some concluding remarks.

We need to introduce some useful notations. We will use the well-known Vallee-Poussin sums, which play a very important role in the approximation theory, see e.g. [9], [54], chapter 5.

Recall that the Vallee-Poussin kernel $K_{n,p}(t)$ is defined as follows:

$$K_{n,p}(t) = \frac{\sin((2n+1-p)t/2) \cdot \sin((p+1)t/2)}{2(p+1) \sin^2 t/2}, \quad p \in [1, 2, \dots, n].$$

It is known that $K_{n,p}(t)$ is trigonometrical polynomial of degree n : $K_{n,p}(t) \in A(n)$, where $A(n)$ denotes the set of all 2π periodical trigonometrical polynomials of degree $\leq n$.

The Vallee-Poussin approximation (sum) $V_{n,p}[f](t)$ for a periodical integrable function f may be defined as follows:

$$V_{n,p}[f](t) := [f * K_{n,p}](t)$$

(periodical convolution). Recall that

$$V_{n,p}[f](t) = \frac{1}{p+1} \sum_{k=n-p}^n S_k[f](t),$$

where $S_k[f](t)$ is the k^{th} partial Fourier sum for the function f .

We pick hereafter for definiteness for the values $n \geq 4$ $p = p(n) \stackrel{def}{=} \text{Ent}(n/2) := [n/2]$, (integer part), so that $V_{n,p}[f](t) \in A(n)$ and

$$||f(\cdot) - V_{n,p(n)}[f](\cdot)|| \leq C \cdot E([n/2], f),$$

where $E(m, f)$ denotes the error in the uniform norm of the best trigonometrical approximation of the function f by means of trigonometrical polynomials with degree $\leq m$:

$$E(m, f) = \inf_{g \in A(m)} \max_{t \in T} |f(t) - g(t)| = \inf_{g \in A(m)} ||f(t) - g(t)||,$$

see [29], chapter 6.

We define for arbitrary r.p. $\xi(t)$ the so-called *generating functional* $\Phi(\xi; \psi)$ as follows:

$$\Phi_\xi = \Phi(\xi; \psi) = \mathbf{E} e^{\int_T \xi(s) d\psi(s)}, \quad (1.2)$$

if there exists. Here $\psi(s)$, $s \in [0, 2\pi]$ is any *deterministic* function of bounded variation.

Further, we denote by \mathcal{N} a set of all strictly increasing sequences of natural numbers $\{n(k)\}$, $k = 1, 2, \dots$, $n(1) = 1$; and define for each such a sequence $\vec{n} = \{n(k)\} \in \mathcal{N}$

$$W^{(k)}(t) = W_{\vec{n}}(n(k+1), n(k))(t) := V_{n(k+1), p(n(k+1))}(t) - V_{n(k), p(n(k))}(t), \quad (1.3)$$

and we define for arbitrary periodical random process $\xi = \xi(t)$, $t \in T$

$$\begin{aligned} Z_k[\xi](t) &= Z_{\vec{n}, k}[\xi](t) = [W^{(k)} * \xi](t) = \\ &= [V_{n(k+1), p(n(k+1))} * \xi](t) - [V_{n(k), p(n(k))} * \xi](t), \end{aligned} \quad (1.4)$$

$$\Psi_{\vec{n}}(\xi; \lambda, n(k), n(k+1)) = \Psi(\xi; \lambda, n(k), n(k+1)) =$$

$$(2\pi)^{-1} \int_T \mathbf{E} e^{\lambda Z_k[\xi](t)} dt, \quad \lambda = \text{const} > 0;$$

evidently, $\Psi_{\vec{n}}(\cdot; \cdot, \cdot, \cdot)$ may be easily expressed through the generating functional $\Phi(\cdot)$. Namely, let

$$\psi_t^{(k)}(s) = \int_0^s W^{(k)}(t-x) dx,$$

then

$$\begin{aligned} \Psi(\xi; \lambda, n(k), n(k+1)) &= (2\pi)^{-1} \int_T \mathbf{E} e^{\lambda \int_T \xi(s) W^{(k)}(t-s) ds} dt = \\ (2\pi)^{-1} \int_T dt \mathbf{E} e^{\lambda \int_T \xi(s) d\psi_t^{(k)}(s)} &= (2\pi)^{-1} \int_T \Phi(\xi; \lambda \psi_t^{(k)}(\cdot)) dt. \end{aligned} \quad (1.5)$$

Define also at last

$$\begin{aligned} U(\Phi_\xi; n(k), n(k+1)) &= U_{\vec{n}}(\xi; n(k), n(k+1)) = \\ \inf_{\lambda > 0} \left[\frac{\log n(k+1) + \log \Psi_{\vec{n}}(\xi; \lambda, n(k), n(k+1))}{\lambda} \right]. \end{aligned} \quad (1.6)$$

2 Main result: sufficient conditions for CLT for continuous processes.

A. Weak compactness of a family of random processes.

Let $\xi_\alpha(t)$, $\alpha \in \mathcal{A}$ be a *family* of separable stochastically continuous periodical processes, \mathcal{A} be arbitrary set. Assume that for some non-random point $t_0 \in T$ the family of one-dimensional r.v. $\xi_\alpha(t_0)$ is tight.

Theorem 2.1.

1. Let $\alpha \in \mathcal{A}$ be a given. If for some sequence $\{n(k)\} \in \mathcal{N}$

$$\sum_{k=1}^{\infty} U(\Phi_{\xi_\alpha}; n(k), n(k+1)) < \infty, \quad (2.1)$$

then almost all trajectories of r.p. $\xi_\alpha(t)$ are continuous.

2. Suppose that there exists a single sequence $\{n(k)\}$ such that for all the set \mathcal{A} the series (2.1) are uniform equiconvergent:

$$\lim_{m \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \sum_{k=m}^{\infty} U(\Phi_{\xi_\alpha}; n(k), n(k+1)) = 0. \quad (2.2)$$

Then the family of distributions $\mu_\alpha(\cdot)$ in the space $C(T)$ generated by $\xi_\alpha(\cdot)$:

$$\mu_\alpha(A) = \mathbf{P}(\xi_\alpha(\cdot) \in A)$$

is weakly compact in this space.

Proof.

Let $\alpha \in \mathcal{A}$ be a fix. We will use one of the main results of the article [43]:

$$\mathbf{E}||Z_k[\xi_\alpha]|| \leq C U(\Phi_{\xi_\alpha}; n(k), n(k+1)), \quad (2.3)$$

where C is an absolute constant.

We conclude by virtue of condition (2.1) that the following series converges:

$$\sum_{k=1}^{\infty} \mathbf{E}||Z_k[\xi_\alpha]|| < \infty,$$

with him

$$\sum_{k=1}^{\infty} ||Z_k[\xi_\alpha]|| < \infty \pmod{\mathbf{P}}.$$

Therefore, the partial sums of $Z_k[\xi_\alpha](t)$, namely, the sequence of the r.p.

$$\sum_{m=1}^k Z_m[\xi_\alpha](t) = [V_{n(k+1), p(n(k+1))} * \xi](t) \quad (2.4)$$

converges uniformly on t , $t \in T$ also with probability one. The limiting as $k \rightarrow \infty$ process coincides with $\xi_\alpha(t)$ since it is continuous in probability; thus, it is sample part continuous. We proved the first proposition of theorem 1.1.

Further, let the condition (2.2) be satisfied. We denote

$$\epsilon(m) = \sup_{\alpha \in \mathcal{A}} \sum_{k=m}^{\infty} U(\Phi_{\xi_\alpha}; n(k), n(k+1)), \quad (2.5)$$

then $\lim_{m \rightarrow \infty} \epsilon(m) = 0$.

As long as

$$\sum_{m=1}^k Z_m[\xi_\alpha](t) = [V_{n(k+1), p(n(k+1))} * \xi](t)$$

is as the function on the variable t the trigonometrical polynomial of degree less than $n(k+1)$, we conclude

$$\sup_{\alpha \in \mathcal{A}} \mathbf{E} E(n(k+1), \xi_\alpha) \leq C_1 \epsilon(n(k)) \rightarrow 0, \quad k \rightarrow \infty. \quad (2.6)$$

We can use the *inverse* theorems of approximation theory; see for example, Stechkin's estimate ([54], chapter 6, section 6.1:)

$$\omega(f, 1/n) \leq \frac{C_2}{n} \cdot \sum_{m=0}^n E_m(f).$$

We ensue for the module of continuity $\omega(\xi_\alpha(\cdot), 1/n)$ for periodical r.p. $\xi_\alpha(t)$:

$$\sup_{\alpha \in \mathcal{A}} \mathbf{E} \omega(\xi_\alpha(\cdot), 1/n) \leq \frac{C_3}{n} \cdot \sum_{m=0}^n \mathbf{E} E_m(\xi).$$

Note as a consequence taking into account the monotonicity of the function $m \rightarrow E(m, f)$:

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \mathbf{E} \omega(\xi_\alpha(\cdot), 1/n) = 0 \quad (2.7)$$

and therefore

$$\forall \epsilon > 0 \Rightarrow \lim_{\delta \rightarrow 0+} \sup_{\alpha \in \mathcal{A}} \mathbf{P}(\omega(\xi_\alpha(\cdot), \delta) > \epsilon) = 0.$$

Our proposition follows now from theorem 1 in the book of I.I.Gikhman and A.V.Skorohod [17], chapter 9, section 2; see also [46], after applying the Tchebychev's inequality.

Remark 2.1. The conditions of theorem 2.1 are essentially non-improvable still for the Gaussian processes, see [12], [27], [40], [41], [42], [43].

B. CLT for random processes.

Theorem 2.2.

Suppose that there exists a single strictly increasing sequence $\{s(k)\}$ of natural numbers such that the following series (2.8) are uniform equiconvergent:

$$\lim_{m \rightarrow \infty} \sup_n \sum_{k=m}^{\infty} U\left(\Phi_{\eta, \psi(\cdot)/\sqrt{n}}^n; s(k), s(k+1)\right) = 0. \quad (2.8)$$

Then the r.p. $\eta(t)$, $t \in T$ satisfies CLT in the space $C(T)$.

Proof. The generating functional for the r.p. $\zeta_n(\cdot)$, i.e. $\Phi_{\zeta_n}(\psi)$ may be calculated and uniformly estimated as follows:

$$\Phi_{\zeta_n}(\psi) = \Phi_{\eta}^n(\psi/\sqrt{n}), \quad (2.9)$$

$$\Phi_{\zeta_n}(\psi) \leq \sup_n \left[\Phi_{\eta}^n(\psi/\sqrt{n}) \right] < \infty.$$

It remains to apply the second assertion of theorem 2.1, choosing $\{\alpha\} = \{1, 2, \dots\}$.

3 Necessary conditions for CLT for continuous processes.

In this section the set T be instead $[0, 2\pi]$ the arbitrary compact metric space equipped with the distance function $d = d(s, t)$. The r.p. $\eta(t)$ remains to be separable and centered.

N.C.Jain and M.B.Marcus in [22] are formulated and proved in particular the following famous result (we retell its in our notations).

Theorem of Jain and Marcus. Assumptions: there exists a random variable M such that

$$|\eta(t) - \eta(s)| \leq M \cdot \rho(t, s), \quad (3.1)$$

condition of factorization; where $\rho(t, s)$ is continuous deterministic distance, more exactly, semi-distance, possible different on the source distance d on the set T ;

$$\mathbf{E}M^2 < \infty, \quad (3.2)$$

moment condition;

$$\int_0^1 H^{1/2}(T, \rho, z) dz < \infty, \quad (3.3)$$

entropy condition. Here $H(\rho, T, \epsilon)$ denotes a metric entropy function of the set T relative the distance $\rho(\cdot, \cdot)$ at the point ϵ , $0 < \epsilon < 1$.

Then the random process (field) $\eta(t)$ satisfies the CLT in the space $C(T, d)$.

We discuss in this section the necessity of all conditions 3.1; 3.2; 3.3 for the CLT in the space $C(T, d)$.

So, it will be presumed sometimes further that the r.p. $\eta(t)$ satisfies this CLT.

Proposition 3.1. Assume that the r.p. $\eta(t)$ is sample part continuous, still without CLT. Then the condition of factorization (3.1) is satisfied.

This assertion follows immediately from the main result of the articles [7], [44]; see also [8].

Proposition 3.2. Suppose in addition $\eta(t)$ satisfies the CLT in $C(T, d)$; then in the factorization (3.1) the metric $\rho(\cdot, \cdot)$ may be selected such that

$$\forall p \in (0, 2) \Rightarrow \mathbf{E}|M|^p < \infty. \quad (3.4)$$

Proof. It is proved in the famous book of M.Ledoux and M.Talagrand [25], chapter 10, section 1, p. 274-277 that if the r.p. $\eta(\cdot)$ satisfies the CLT in the separable Banach space X with the norm $\|\cdot\|_X$, then $\mathbf{E}\|\eta\|^p < \infty$. Therefore

$$\mathbf{E} \sup_{t \in T} |\eta(t)|^p < \infty, \quad p \in (0, 2).$$

Note that the function $u \rightarrow |u|^p$ is Young - Orlicz function satisfying the well-known Δ_2 condition.

It remains to use on of the main result of aforementioned article [44].

Analogously may be proved the following assertion.

Proposition 3.3. Assume that the r.p. $\eta(t)$ is sample part continuous, still without CLT. Moreover, let

$$\mathbf{E} \sup_{t \in T} |\eta(t)|^2 < \infty.$$

Then the distance ρ in the factorization representation (3.1) may be selected such that the r.v. M has finite second moment: $\mathbf{E}M^2 < \infty$.

Before formulating the next result, we introduce together with R.M.Dudley [11], N.C.Jain and M.B.Marcus [22] the third distance $\tau = \tau(t, s)$ on the set R , also natural, as follows:

$$\tau(t, s) = \sqrt{\text{Var}(\eta(t) - \eta(s))} = [\mathbf{E}(\eta(t) - \eta(s))^2]^{1/2}. \quad (3.5)$$

It follows from the condition (3.1) that $\tau(t, s) \leq C \cdot \rho(t, s)$.

Proposition 3.4. Suppose the r.p. $\eta(t)$, $t \in [0, 2\pi]$ satisfies the CLT in $C(T, d)$, is stationary in wide sense. Assume also that the metric functions $\rho(r, s)$ and $\tau(t, s)$ are linear equivalent:

$$C_1 \rho(t, s) \leq \tau(t, s) \leq C_2 \rho(t, s), \quad C_1, C_2 = \text{const} \in (0, \infty). \quad (3.6)$$

Then the entropy condition (3.3) is satisfied.

Proof. Since the limit process $\zeta_\infty(t)$ is continuous, Gaussian and has at the same covariation function as $\eta(t)$, it is stationary in strong sense and also continuous. Our assertion follows from the famous necessary condition for sample part continuity of Gaussian stationary process belonging to X.Fernique [12].

Remark 3.1. At the same result is true even without assumption of stationarity with at the same proof if the function $t \rightarrow \tau(t, 0)$ is monotonic in some neighborhood of zero, see [27].

4 Some examples.

1. Let here $T = [0, e^{-4}]$ and let $\delta = \text{const} \in (0, 1/4)$. We define the r.p. $\eta_0(t)$, $t \in T$ as follows.

$$\eta_0(t) := \frac{w(t)}{(2t)^{1/2} (\log |\log t|)^{1/2+\delta/2}}, \quad 0 < t \leq e^{-4}, \quad (4.1)$$

and $\eta_0(0) := 0$; $w(t)$ is ordinary Brownian motion.

It follows from the classical Law of Iterated Logarithm (LIL) for Wiener process that the r.p. $\eta_0(t)$ is continuous a.e. Since it is "per se" Gaussian, it satisfies the CLT($C(T)$). The conditions (3.1) and (3.2) are also satisfied, but the entropy integral (3.3) is divergent.

Indeed, we have denoting

$$\tau_0(t, s) = \sqrt{\text{Var}(\eta_0(t) - \eta_0(s))} :$$

$$\tau_0(t, 0) \sim \frac{C}{[\log |\log t|]^{1+\delta}},$$

therefore

$$h_+(\epsilon) := \sup_{t \in T} \mu(B(t, \epsilon)) \geq \mu(B(0, \epsilon)) \geq C \cdot \exp \left(\exp \left(C_1 \epsilon^{1/(1+\delta)} \right) \right), \quad 0 < \epsilon < 1/8, \quad (4.2)$$

where

$$B(t, \epsilon) = \{s : s \in T, \tau_0(t, s) \leq \epsilon\}$$

and μ is ordinary Lebesgue measure on the real axis.

We can apply the following inequality, see [30], chapter 3, section 3.2:

$$\exp H(T, \tau_0, \epsilon) \geq \frac{\mu(T)}{h_+(\epsilon)},$$

or equally

$$H(T, \tau_0, \epsilon) \geq \exp \left(C_2 \epsilon^{1/(1+\delta)} \right). \quad (4.3)$$

It is easy to verify that for such a entropy the condition (3.3) is not satisfied.

2. The following example appears in the Monte-Carlo method for computation and error estimation of multiple multivariate parametric integrals, see [15], [19], [11], [30], [37] etc. Namely, let us consider the problem of computation of the following parametric integral

$$I(t) = \int_D v(t, x) \mu(dx),$$

where μ is probability measure: $\mu(D) = 1$, by means of Monte-Carlo method:

$$I(t) \approx I_n(t) := n^{-1} \sum_{j=1}^n v(t, \beta_j),$$

where $\{\beta_j\}$ are independent r.v. with distribution μ .

Consider for error estimation the following variable:

$$\gamma_n(u) := \mathbf{P} \left(\sqrt{n} \sup_t |I_n(t) - I(t)| > u \right).$$

We put here $\eta_I(t) = v(t, \beta_1) - I(t)$. If $\eta_I(t)$ satisfies the CLT(C(T)), then as $n \rightarrow \infty$

$$\gamma_n(u) \rightarrow \mathbf{P} \left(\max_{t \in T} |\zeta_\infty(t)| > u \right),$$

therefore

$$\gamma_n(u) \approx \mathbf{P} \left(\max_{t \in T} |\zeta_\infty(t)| > u \right) =: \gamma_\infty(u).$$

The asymptotical behavior or exact estimations as $u \rightarrow \infty$ for the right-hand side of the last equality are well known, see [30], chapter 3; [45]. As a rule as $u \rightarrow \infty$

$$\gamma_\infty(u) \sim K \cdot u^{\kappa-1} \cdot \exp \left(-\frac{u^2}{2\sigma^2} \right),$$

where

$$\sigma^2 = \max_{t \in T} \text{Var } \eta(t) = \max_{t \in T} R(t, t), \quad K, \kappa = \text{const} \in (0, \infty).$$

Let ε be some "small" number, for instance, 0.05 or 0.01 etc. Denote by $U(\varepsilon)$ the maximal root of equation

$$\gamma_\infty(U(\varepsilon)) = \varepsilon,$$

we deduce that if $\eta(t)$ satisfies CLT(C(T)), then with probability $\approx 1 - \varepsilon$

$$\sup_{t \in T} |I_n(t) - I(t)| \leq \frac{U(\varepsilon)}{\sqrt{n}};$$

which is a twice asymptotical: as $n \rightarrow \infty$ and as $u \rightarrow \infty$ confidence region for $I(t)$ in the uniform norm.

The non-asymptotical exponential exact estimation for $\gamma_\infty(u)$ in the modern terms of majorizing measures see in the article [33].

Suppose there exist a r.v. θ and a non-random monotonically decreasing sequence $\delta(n)$, such that

$$E(n, g) \leq \theta \cdot \delta(n), \quad \lim_{n \rightarrow \infty} \delta(n) = 0. \quad (4.4)$$

We impose the following condition on the r.v. θ :

$$\exists m = \text{const} > 1, \quad C = \text{const} \in (0, \infty), \quad \mathbf{P}(\theta > x) \leq e^{-Cx^m}, \quad x \geq 0, \quad (4.5)$$

or equally

$$|\theta|_p \leq C_2 p^{1/m}, \quad p \geq 1,$$

and denote $\tilde{m} = \min(m, 2)$, $m' = \tilde{m}/(\tilde{m} - 1)$. The conditions of theorem 2.2 are satisfied if for example

$$\sum_{r=1}^{\infty} \frac{\delta(2^r)}{r^{1/\tilde{m}}} < \infty. \quad (4.6).$$

The condition (4.6) is satisfied in turn if for instance for some positive value $\Delta = \text{const} > 0$

$$\delta(n) \leq \frac{C_3}{[\log(n+2)]^{1/m'+\Delta}}. \quad (4.7)$$

We used the following proposition, see [23], [30], chapter 2, section 2.1, page 50-53: if ξ_i , $i = 1, 2, \dots$ are centered, i.e., i.i.d. r.v. such that

$$\mathbf{P}(|\xi_i| > x) \leq \exp(-x^m), \quad x > 0,$$

then

$$\sup_n \mathbf{P} \left(n^{-1/2} \left| \sum_{i=1}^n \xi_i \right| > x \right) \leq \exp(-C(m) x^{\tilde{m}}), \quad x > 0,$$

and the last inequality is exact.

The examples of a (random) functions $g(\cdot)$ for which

$$\delta(n) \asymp \frac{C_4}{[\log(n+2)]^{1/m'+\Delta}}$$

may be constructed by means of lacunary random series, see [28], [43].

Notice that the conditions (4.6) and (4.7) are so weak so that the entropy condition [22] and majorizing measures conditions [20] are not satisfied.

Remark 4.1. The condition (4.4) is satisfied if for example

$$\omega(g, \epsilon) := \sup_{h: |h| \leq \epsilon} \|g(\cdot + h) - g(\cdot)\| \leq C \cdot \theta \cdot \delta(\text{Ent}[1/\epsilon]),$$

Jackson's inequality; see [1], [10], [26], [29]; but the inverse inequality is non true, see aforementioned Stechkin's estimate ([54], chapter 6, section 6.1).

3. Let now the set $T = \{1, 2, \dots; \infty\}$ equipped with the distance

$$d(i, j) = \left| \frac{1}{j} - \frac{1}{i} \right|, \quad d(i, \infty) = \frac{1}{i}, \quad d(\infty, \infty) = 0. \quad (4.8)$$

The set T is compact set relatively the distance d .

We choose in the capacity of probability space the ordinary interval $(0, 1)$ with Lebesgue measure.

Let $p_0 = \text{const} \in (1, 2)$,

$$\alpha = \text{const} \in (0, \min(1, p_0/(2 - p_0))), \quad a(n) = 1 - 0.5n^{-\alpha}, \quad n = 1, 2, \dots;$$

$$\Delta(n) = a(n+1) - a(n), \quad c(n) = n^{\alpha/p_0}.$$

Let also

$$f_{1/2}(x) = \sqrt{|\log x|}, \quad x \in (0, 1); \quad f_{1/2}(x) = 0, \quad x \notin (0, 1). \quad (4.9)$$

We define for the values $n \in T$ the following r.p.

$$\eta_n(x) = c(n) \epsilon_n f_{1/2} \left(\frac{x - a(n)}{\Delta(n)} \right), \quad \eta_\infty(x) = 0, \quad (4.10)$$

where $\{\epsilon_n\}$ is a Rademacher's sequence defined on some other probabilistic space and independent on $f_{1/2}(\cdot)$, so that $\mathbf{E}\eta_n = 0$.

Recall that

$$\|\eta(\cdot)\| = \sup_n |\eta_n|.$$

This example was offered by the authors in [35] for another purpose. It was proved in particular in [35] that the r.p. η_n is continuous with probability one, this imply:

$$\mathbf{P} \left(\lim_{n \rightarrow \infty} \eta_n \rightarrow 0 \right) = 1,$$

and

$$\mathbf{E}||\eta||^{p_0} < \infty, \forall p > p_0 \Rightarrow \mathbf{E}||\eta||^p = \infty. \quad (4.11)$$

Let us prove in addition that the sequence of a r.v. $\{\eta_n\}$ is pre-gaussian in the considered space $C(T, d)$. This imply by definition that the mean zero Gaussian distributed sequence $\{\nu_n\}$ with at the same covariation function $Q(n_1, n_2)$ as $\{\eta_n\}$ belong to the set $C(T, d)$ with probability one.

Namely, since the Rademacher's sequence contains from independent r.v. ϵ_n ,

$$Q(n_1, n_2) = 0, \quad n_1 \neq n_2.$$

It suffices to prove $\mathbf{P}(\nu_n \rightarrow 0) = 1$. We have:

$$\text{Var } \nu_n = Q(n, n) = \text{Var } \eta_n = \mathbf{E}|\eta_n|^2.$$

It is proved in [35] that

$$\mathbf{E}|\eta_n|^2 \leq C_1 n^{-C_2}, \quad C_2 > 0,$$

therefore $\text{Var } \nu_n \leq C_1 n^{-C_2}$ and by virtue of Gassiness of the sequence $\{\nu_n\}$

$$\forall \epsilon > 0 \Rightarrow \sum_{n=1}^{\infty} \mathbf{P}(|\nu_n| > \epsilon) < \infty.$$

Thus,

$$\mathbf{P}\left(\lim_{n \rightarrow \infty} \nu_n \rightarrow 0\right) = 1.$$

The r.p. η_n does not satisfy the majorizing measure condition, [35]. It follows from (4.11) that the condition (3.4) is also not satisfied. Therefore, η_n does not satisfy CLT in our space $C(T, d)$.

5 Concluding remarks.

1. Note that the r.v. M and the distance $\rho = \rho(t, s)$ in (3.1) may be introduced constructively in two stages as follows. First step: define the r.v. L

$$L := \sup_{t, s \in T} |\eta(t) - \eta(s)|.$$

Second step:

$$q(t, s) := \text{vraisup}_{\Omega} \left[\frac{|\eta(t) - \eta(s)|}{L} \right];$$

(the case $L = 0$ is trivial). Then the function $(t, s) \rightarrow q(t, s)$ is continuous bounded distance (more exactly, semi-distance) on the set T and

$$|\eta(t) - \eta(s)| \leq L \cdot q(t, s).$$

Obviously, this choice of the variables L, q is optimal.

2. It is clear that we can use instead trigonometrical approximation the approximation by means of algebraic polynomials.

3. Suppose that instead the condition (3.6) is true the following inequality:

$$\rho(t, s) \leq C \cdot \tau^{1/\beta}(t, s), \quad \beta = \text{const} \in (0, 1).$$

Then the *necessary* condition of a view (3.3) must be transformed as follows:

$$\int_0^1 H^{1/2}(T, \rho, z^\beta) dz < \infty.$$

4. It is no hard to generalize our results into the multidimensional case $T = [0, 2\pi]^d$ and into the non-periodical continuous function space.

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